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Total Positivity, Spherical Series, and Hypergeometric Functions of Matrix Argument*

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Given a totally positive function K of two real variables, is there a method for establishing the total positivity of K in an "obvious" fashion? In the case in which $K(x, y) = f(xy)$, where f is real-analytic in a neighborhood of zero, we obtain integral representations for the determinants which define the total positivity of K. The total positivity of K then follows immediately from positivity of the integrands. In particular, we analyze the total positivity of classical hypergeometric functions by these methods. The central theme of this work is the circle of ideals that relates total positivity to "spherical series" on the symmetric space $GL(n, \mathbf{C})/U(n)$, and classical hypergeometric functions to hypergeometric functions of matrix argument.

INTRODUCTION

This paper is an outgrowth of our work [5] and [6] on the theory of special functions of matrix argument. Here, we are concerned with applications to the theory of totally positive functions on the real line. More specifically, let f be a real-analytic function of a real variable and set $K(x, y) = f(xy)$. We utilize harmonic analysis on the space S_n of $n \times n$ Hermitian matrices to establish necessary and sufficient criteria for K to be totally positive.

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In order to give some of the flavor of our ideas, we will relate briefly the way in which we were led to the present applications. In [5], we developed the theory of generalized hypergeometric functions of matrix argument over real division algebras. In direct analogy to the definition of classical hypergeometric functions as power series, hypergeometric functions of matrix argument are defined as *spherical series*; that is, infinite series expansions in *spherical functions* (otherwise known as *zonal polynomials*). Then, in [6], we explored the fine structure of such functions over the complex field. In the course of that investigation we evaluated the hypergeometric functions of matrix argument in terms of determinants of related classical hypergeometric functions. Subsequently, in considering the inverse problem, we realized that our methods applied much more generally to relating determinants of real-analytic functions of a real variable to spherical series.

Our approach to total positivity rests on an integral formula (see Theorem 3.1 and Corollary 3.2) which we now describe. Let f and K be as above, and let $U(n)$ denote the group of $n \times n$ unitary matrices. To the function f we associate a sequence of functions ψ_n on S_n such that for $n = 1, 2, \ldots$,

$$\frac{\det(K(s_i, t_j))}{V(s)\, V(t)} = \int_{U(n)} \psi_n(sutu^{-1})\, du.$$

Here, s and t are matrices in S_n having eigenvalues $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$, respectively, and for which all products $s_i t_j$ lie in the domain of f; and $V(s) = \prod_{1 \leqslant i < j \leqslant n} (s_i - s_j)$ denotes the Vandermonde determinant. It follows immediately from the definition of total positivity that K is totally positive if and only if the functions ψ_n are non-negative.

In principle, the above integral formula gives precise necessary and sufficient conditions for total positivity. In practice, however, the functions ψ_n are defined by spherical series expansions, a mode of presentation not very well suited to expressing the non-negativity of ψ_n. Nonetheless, when f is a classical hypergeometric function the situation is quite tractable; for in that context we prove (see Theorem 4.2 and Corollary 4.3) that the functions ψ_n are hypergeometric functions of matrix argument for which the theory in [5] and [6] applies. More specifically, a classical hypergeometric function $K(x, y) = {}_p\mathscr{F}_q(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ xy)$ is totally positive of order r if and only if the corresponding hypergeometric functions $\psi_n(t) = {}_pF_q(a_1 + n - 1, \ldots, a_p + n - 1;\ b_1 + n - 1, \ldots, b_q + n - 1;\ t)$ of matrix argument are non-negative for all $n = 1, 2, \ldots, r$. Perhaps this result will stimulate further study of the non-negativity of hypergeometric functions of matrix argument.

At the close of these introductory remarks, it is appropriate to comment that our work is not the first to relate determinants of real-analytic functions to spherical series. Indeed, the treatise [8] by L.-K. Hua on complex analysis on classical domains was a source of great inspiration for our work on total positivity. In particular, our integral formula above was motivated by Hua's masterful calculations of spherical series. It is a pleasure for us to record our admiration of Hua's work.

1. Totally Positive Functions

Let $\mathscr{D}$ be a subset of $\mathbf{R}^2$ and r be a positive integer. A function $K: \mathscr{D} \to \mathbf{R}$ is *totally positive of order* r (TP_r) if for all $s_1 < s_2 < \cdots < s_r$ and $t_1 < t_2 < \cdots < t_r$ such that $(s_i, t_j) \in \mathscr{D}$, the $n \times n$ determinant

$$\det(K(s_i, t_j)) = \begin{vmatrix} K(s_1, t_1) & K(s_1, t_2) & \cdots & K(s_1, t_n) \\ K(s_2, t_1) & K(s_2, t_2) & \cdots & K(s_2, t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n, t_1) & K(s_n, t_2) & \cdots & K(s_n, t_n) \end{vmatrix} \tag{1.1}$$

is non-negative for all $n = 1, ..., r$. If the determinants $\det(K(s_i, t_j))$ are all strictly positive, then we say that K is *strictly totally positive of order* r (STP_r). If K is TP_r (STP_r) for all $r = 1, 2, ...$, then we say that K is TP_∞ (STP_∞).

The principal reference for the subject of total positivity is Karlin's comprehensive treatise [10]. In statistics, the TP_2 property is known as the *monotone likelihood ratio property* [10, 14]. Other references to the subject of total positivity are [1, 11, 12, and 16]. The theory of totally positive functions interfaces in significant ways with many areas of probability theory, statistics, mathematical analysis, and mathematical physics. In some instances TP_∞ implies positive definiteness, and then the TP_∞ function determines a reproducing kernel space. We refer to Chapter I of [10] for a survey of some of these applications. The reader might also consult [1, 2, 3, 11, 12, 14, and 16] for further applications. The connection between reproducing kernels and total positivity is exploited in [2, 3]. The starting point for our work is the observation that the most widely studied classes of totally positive functions arise from hypergeometric functions.

This paper involves a new approach to some basic questions. Given a function K, how do we verify that K is TP_r for some r? That is to say, if K is TP_r is there a method for verifying this property which makes the total positivity of K immediately clear? If K is a non-negative function which is not TP_∞, how can we find the largest r such that K is TP_r? Can we extend the set $\mathscr{D}$ on which K is totally positive?

In regard to these questions, we remind the reader of *Polya's principle*: If an entity K possesses a property $\mathscr{P}$, then there should be a way of viewing K so that the property $\mathscr{P}$ becomes "obvious." In our situation, Polya's principle takes the form of the integral representation (3.8) alluded to in the introduction.

We illustrate our ideas with two examples, $K_1(x, y) = e^{xy}$ and $K_2(x, y) = (1-xy)^{-a}$, $a > 0$. That K_1 is STP_∞ on $\mathscr{D} = \mathbf{R}^2$ is proved in [10, pp. 15–16] by counting the zeros of a certain exponential polynomial. An alternative proof by induction is also outlined in [14, p. 119]. In the case of K_2, the power series arguments in [10, p. 101] readily establish the STP_∞ property on $(0, 1) \times (0, 1)$. By our methods, all of these results for K_1 and K_2 follow immediately from the integral representation (3.8) (cf. Corollaries 3.4 and 3.5). In addition, the integral formula easily implies that K_2 is STP_∞ on the larger domain $\{(x, y) \in \mathbb{R}^2\colon |xy| < 1\}$. Moreover, for no $\rho > 1$ is K_2 totally positive on $\{(x, y) \in \mathbf{R}^2\colon |xy| < \rho\}$.

It should be mentioned that, in practice, the integral formula typically implies a stronger notion of total positivity [10, Chap. 2, Section 2]. A function K is said to be *extended totally positive* (ETP) on $\mathscr{D}$ if for all $n = 1, 2, \ldots$, the $n \times n$ determinant

$$\tilde{K}(x, y) = \det\left(\frac{\partial^{i+j-2}}{\partial x^{i-1}\,\partial y^{j-1}} K(x, y)\right) \tag{1.2}$$

is positive for all $(x, y) \in \mathscr{D}$. By repeated applications of the mean-value theorem, it can be shown that if K is ETP then it is STP_∞. Conversely, if K is STP_∞ and real-analytic, then it is ETP. In our work, we obtain the extended total positivity of K, whenever it is valid, from the integral representation (3.8) by means of the formula [8, Theorem 1.2.4]

$$\tilde{K}(x, y) = \beta_n^2 \lim_{s_i \to x,\, t_j \to y;\, i,j = 1, \ldots, n} \frac{\det(K(s_i, t_j))}{V(s)\, V(t)}, \tag{1.3}$$

where

$$\beta_n = \prod_{j=1}^{n} (j-1)!. \tag{1.4}$$

2. Schur Functions and Spherical Series

We briefly outline the harmonic analysis needed for this paper.

A *partition* is an n-tuple $m = (m_1, \ldots, m_n)$ of nonnegative integers such that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$. To each partition m is associated an entity called a *Schur function.* There are three distinct ways of viewing Schur functions: (1) Explicitly as symmetric polynomials on $\mathbf{R}^n$; (2) as characters of irreducible holomorphic representations of the general linear group $GL(n, \mathbf{C})$; and (3) as *spherical functions* on the space of $n \times n$ Hermitian matrices.

Explicitly, the *Schur function* χ_m *of index* m is defined on $\mathbf{R}^n$ by the formula

$$\chi_m(t_1, ..., t_n) = \frac{\det(t_i^{m_j+n-j})}{V(t)} \tag{2.1}$$

whenever the real numbers t_i are distinct, and by L'Hôpital's rule otherwise. Then χ_m is a polynomial on $\mathbf{R}^n$, invariant under the symmetric group on n letters, and homogeneous of degree

$$|m| = m_1 + \cdots m_n. \tag{2.2}$$

The connection with character theory is as follows. To a partition m there corresponds an irreducible holomorphic representation λ_m of $GL(n, \mathbf{C})$. *Weyl's character formula* provides the explicit formula [21, p. 200]

$$\operatorname{tr}(\lambda_m(t)) = \chi_m(t_1, ..., t_n) \tag{2.3}$$

for the character of λ_m on the diagonal matrices $t = \operatorname{diag}(t_1, ..., t_n)$ in $GL(n, \mathbf{C})$. Then *Weyl's dimension formula* [21, p. 201]

$$d_m = \chi_m(1, ..., 1) = \beta_n^{-1} \prod_{1 \leqslant j < k \leqslant n} (m_j - m_k - j + k) \tag{2.4}$$

for the degree d_m of λ_m follows from (2.1) in the limit as $t_j \to 1$ for $j = 1, ..., n$. Here, β_n is the constant (1.4). The above relationship between Schur functions and the character theory of $GL(n, \mathbf{C})$ is well known, but it does not directly concern us in this paper. On the other hand, the interplay between Schur functions and spherical functions is perhaps less widely known, but is essential to our work.

Denote by $S = S_n$ the real vector space of all $n \times n$ complex Hermitian matrices $t = t^*$, on which the general linear group $G = GL(n, \mathbf{C})$ of non-singular $n \times n$ matrices acts by $t \to a^*ta$. The *unitary group* $U = U(n) = \{u \in G\colon uu^* = 1\}$ is the stability group of the identity matrix $1 \in S$ and the orbit of $1 \in S$ under G is the open convex cone P in S of positive-definite Hermitian $n \times n$ matrices. The map $Ua \to t = a^*a$ identifies the symmetric space $U \backslash G$ with the cone P.

Let $\mathscr{P}(S)$ be the algebra of all polynomial functions on S, viewed as a module with respect to the group G. A *spherical function* on S is a non-zero element $\phi \in \mathscr{P}(S)$ such that

$$\phi(utu^{-1}) = \phi(t) \tag{2.5}$$

for all $(t, u) \in S \times U$, and whose translates under G span an irreducible subspace of $\mathscr{P}(S)$. The functional equation [5, Proposition 5.5]

$$\int_U \phi(sutu^{-1})\, du = \frac{\phi(s)\, \phi(t)}{\phi(1)}, \tag{2.6}$$

$(s, t) \in S \times S$, is characteristic of spherical functions. Here, du denotes Haar measure on U normalized so that the total volume of U is 1. This *mean-value theorem* plays a key role in our work.

In the statistics literature a spherical function is referred to as a *zonal polynomial.*

To make the connection with Schur functions, note from U-invariance (2.5) that a spherical function is determined uniquely by its restriction to the diagonal matrices. Thus, we may set

$$\phi(t_1, ..., t_n) = \phi(t), \tag{2.7}$$

where $t_1, ..., t_n$ are the eigenvalues of $t \in S$, and view ϕ as a symmetric polynomial on $\mathbf{R}^n$. The classification theorem [5, Proposition 4.11] is as follows. *Up to scalar multiples the spherical functions coincide with the Schur functions.* That is to say, given a spherical function ϕ there exist a partition m and a constant κ such that

$$\phi(t) = \kappa \chi_m(t) \tag{2.8}$$

for all $t \in S$.

The normalizing constant κ in (2.8) is of considerable importance. If we choose $\kappa = d_m^{-1}$ and denote the resulting spherical function by ϕ_m, then $\phi_m(1) = 1$. For this normalization there is a simple estimate [5, p. 805]

$$|\phi_m(t)| \leqslant \|t\|^{|m|}, \tag{2.9}$$

where

$$\|t\| = \max\{|t_i| : i = 1, ..., n\}. \tag{2.10}$$

For our purposes, however, a more useful normalization is given by

$$Z_m(t) = \omega_m \chi_m(t), \tag{2.11}$$

where

$$\omega_m = |m|! \frac{\prod_{1 \leqslant j < k \leqslant n} (m_j - m_k - j + k)}{\prod_{j=1}^{n} (m_j + n - j)!} \tag{2.12}$$

The significance of the normalization (2.11) lies in the expansion of the powers of the trace [5, (5.3.2)] as

$$(\mathrm{tr}\, t)^j = \sum_{|m| = j} Z_m(t) \tag{2.13}$$

for all nonnegative integers j, and the resulting expansion

$$e^{\mathrm{tr}(t)} = \sum_m \frac{1}{|m|!} Z_m(t) \tag{2.14}$$

which converges absolutely for all $t \in S$. In (2.14), the index of summation m ranges over all partitions. Moreover, from (2.14) it is not difficult to derive the *generalized binomial theorem for* $S = S_n$ [5, (6.2.2)]

$$\det(1 - t)^{-a} = \sum_m \frac{[a]_m}{|m|!} Z_m(t) \tag{2.15}$$

which, for any complex number a, converges absolutely for all $t \in S$ such that $\|t\| < 1$. Here, for any $a \in \mathbf{C}$ and partition m we have introduced the *generalized Pochhammer symbol* for S [5, (5.6.3)], defined by

$$[a]_m = \prod_{j=1}^{n} (a - j + 1)_{m_j}, \tag{2.16}$$

where

$$(a)_k = a(a+1) \cdots (a+k-1) \tag{2.17}$$

denotes the usual Pochhammer symbol. In this notation, (2.12) takes the concise form

$$\omega_m = \frac{|m|!\, d_m}{[n]_m}. \tag{2.18}$$

The expansions (2.14) and (2.15) are examples of *spherical series* on S. In general, we define a spherical series as an expansion

$$F(t) = \sum_m b_m Z_m(t) \tag{2.19}$$

of a real-analytic U-invariant function F on S.

3. Integral Representations

In this section we are concerned with the question of total positivity for functions of the form $K(x, y) = f(xy)$ where f is real-analytic. Specifically, we make the connection between spherical series and an integral representation for the determinant (1.1), thereby obtaining a necessary and sufficient criterion for total positivity.

Throughout, we let

$$V(t) = V(t_1, ..., t_n) = \prod_{1 \leqslant i < j \leqslant n} (t_i - t_j) \tag{3.1}$$

be the Vandermonde determinant, where t denotes either a matrix in S or the n-tuple $(t_1, ..., t_n)$ of its eigenvalues, and we set

$$v_m = \frac{d_m}{\omega_m} \tag{3.2}$$

for each partition m. From (2.4) and (2.12) v_m is evaluated as

$$v_m = \frac{1}{|m|!} \prod_{j=1}^{n} \frac{(m_j + n - j)!}{(j-1)!} \tag{3.3}$$

or from (2.18) as

$$v_m = \frac{[n]_m}{|m|!}. \tag{3.4}$$

Finally, if f is a function of a real variable x having a power series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{3.5}$$

then for each positive integer n we define an *associated spherical series* $\psi = \psi_{n,f}$ by

$$\psi(t) = \sum_m A_m Z_m(t), \tag{3.6}$$

where

$$A_m = v_m \prod_{j=1}^{n} a_{m_j + n - j}. \tag{3.7}$$

THEOREM 3.1. *Let f be given by* (3.5) *and assume that its power series converges for* $|x| < \rho$. *Then the spherical series* (3.6) *for* $\psi_{n,f}$ *converges absolutely and defines a real-analytic function* $\psi = \psi_{n,f}$ *on the domain* $\{t \in S_n \colon \|t\| < \rho\}$. *Moreover, if we set* $K(x, y) = f(xy)$ *for* $|xy| < \rho$, *then*

$$\frac{\det(K(s_i, t_j))}{V(s)\, V(t)} = \int_{U(n)} \psi\,(sutu^{-1})\, du \tag{3.8}$$

for all $(s, t) \in S_n \times S_n$ *such that* $\|s\| \cdot \|t\| < \rho$.

Proof. From the absolute convergence of the series (3.5) and the Binet–Cauchy formula [10, p. 1]

$$\det(K(s_i, t_j)) = \sum_{k_1 > \cdots > k_n \geqslant 0} a_{k_1} \cdots a_{k_n} \det(s_i^{k_j}) \det(t_i^{k_j}) \tag{3.9}$$

(cf. [8, Theorem 1.2.2]). Substituting $m_j = k_j - n + j$ for $j = 1, ..., n$, we obtain $m_1 \geqslant \cdots \geqslant m_n \geqslant 0$. The right-hand side of (3.15) then becomes

$$\sum_m a_{m_1+n-1} a_{m_2+n-2} \cdots a_{m_n} \det(s_i^{m_j+n-j}) \det(t_i^{m_j+n-j}) \tag{3.10}$$

which is absolutely convergent for $|s_i t_j| < \rho$. We introduce the definitions (2.1), (2.11), (3.1), (3.3), and (3.7) into (3.10) to obtain

$$\frac{\det(K(s_i, t_j))}{V(s)\, V(t)} = \sum_m A_m \frac{Z_m(s)\, Z_m(t)}{Z_m(1)}. \tag{3.11}$$

The right-hand side of (3.11) converges absolutely on the domain $\{(s, t) \in S \times S \colon \|s\| \cdot \|t\| < \rho\}$ and determines a real-analytic function

$$\xi(s, t) = \frac{\det(K(s_i, t_j))}{V(s)\, V(t)} \tag{3.12}$$

on that domain. The function ψ defined by (3.6) is just the cross-section $\psi(t) = \xi(1, t)$ at $s = 1$. Now we apply the mean-value theorem (2.6) to the right side of (3.11) to obtain

$$\begin{aligned} \frac{\det(K(s_i, t_j))}{V(s)\, V(t)} &= \sum_m A_m \int_{U(n)} Z_m(sutu^{-1})\, du \\ &= \int_{U(n)} \Big(\sum_m A_m Z_m(sutu^{-1}) \Big)\, du \\ &= \int_{U(n)} \psi(sutu^{-1})\, du \end{aligned}$$

and complete the proof.

In short, the theorem associates to each function f on $\mathbf{R}$ that is real-analytic at the origin, or more properly to the corresponding *kernel* $K(x, y) = f(xy)$, a sequence of functions $\psi = \psi_{n,f}$ that are real-analytic on a neighborhood of the origin in S_n. In turn, these functions ψ characterize the determinants (1.1), and *a fortiori*—in line with Polya's principle—the total positivity of K.

COROLLARY 3.2. *Let f and K be as in the theorem. The function K is totally positive of order r on the set $\{(x, y) \in \mathbf{R}^2\colon |xy| < \rho\}$ if and only if the functions $\psi_{n,f}$ are non-negative on $\{t \in S_n\colon \|t\| < \rho\}$ for all $n \leqslant r$.*

The proof is an immediate consequence of (3.8) and the definition of total positivity. Note that (3.8) also implies the stronger condition that K is *strictly* totally positive, even *extended* totally positive, on $\{(x, y) \in \mathbf{R}^2\colon |xy| < \rho\}$ provided that ψ does not vanish identically on any $U(n)$-orbit in $\{t \in S_n\colon \|t\| < \rho\}$.

COROLLARY 3.3. *If the coefficients a_k in the power series (3.5) are non-negative, then K is STP_∞ on the domain $\{(x, y) \in \mathbf{R}^2\colon x > 0,\ y > 0,\ xy < \rho\}$.*

Corollary 3.3 follows directly from the fact that $Z_m(t) > 0$ for all $t \in P$. For the functions ψ_n are then all seen to be non-negative on the indicated domain. This result is also proved in [10, Chap. 2] by more classical means.

To illustrate the application of Corollary 3.2, by means of the integral formula (3.8) we derive the total positivity of the functions K_1 and K_2 that were considered in Section 1.

COROLLARY 3.4. *The function $K_1(x, y) = e^{xy}$ is STP_∞ on $\mathbf{R}^2$.*

Proof. Let $f(x) = e^x$ in Theorem 3.1. From the power series for f, (3.3), and (3.7), the corresponding function $\psi(t)$ is given by (3.6) with

$$A_m = v_m \prod_{j=1}^{n} \frac{1}{(m_j + n - j)!} = \frac{\beta_n^{-1}}{|m|!}, \tag{3.13}$$

where β_n is the constant (1.5). Thus, by (2.14)

$$\psi(t) = \beta_n^{-1} \sum_m \frac{1}{|m|!} Z_m(t) = \beta_n^{-1} e^{\operatorname{tr} t} \tag{3.14}$$

for all $t \in S$. By (3.8)

$$\frac{\det(K_1(s_i, t_j))}{V(s)\, V(t)} = \beta_n^{-1} \int_{U(n)} e^{\operatorname{tr}(sutu^{-1})}\, du. \tag{3.15}$$

Since $e^{\operatorname{tr}(sutu^{-1})} > 0$ for all $s, t \in S$ and $u \in U$, it is obvious that $K_1(x, y) = e^{xy}$ is STP_∞ on $\mathbf{R}^2$.

COROLLARY 3.5. *For $a > 0$, the function $K_2(x, y) = (1 - xy)^{-a}$ is STP_∞ on the domain $\{(x, y) \in \mathbf{R}^2\colon |xy| < 1\}$. More generally, for all $a \in \mathbf{R}$ the sign of* $\det(K(s_i, t_j))$ *agrees with the sign of $\prod_{j=1}^n (a)_{j-1}$.*

Proof. In Theorem 3.1 let $f(x) = (1 - x)^{-a}$ for $|x| < 1$. From the binomial theorem,

$$f(x) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k \tag{3.16}$$

for $|x| < 1$. By (3.3) and (3.7), the coefficients for the corresponding functions $\psi(t)$ are

$$A_m = v_m \prod_{j=1}^{n} \frac{(a)_{m_j+n-j}}{(m_j + n - j)!} = \frac{\beta_n^{-1}}{|m|!} \prod_{j=1}^{n} (a)_{m_j+n-j}. \tag{3.17}$$

Since $(a)_{m_j+n-j} = (a)_{n-j}(a + n - j)_{m_j}$, by (2.15) we obtain

$$A_m = \frac{c_n}{|m|!} \prod_{j=1}^{n} (a + n - j)_{m_j} = c_n \frac{[a + n - 1]_m}{|m|!}, \tag{3.18}$$

where

$$c_n = \beta_n^{-1} \prod_{j=1}^{n} (a)_{j-1}. \tag{3.19}$$

Thus, by (2.15) and (3.6),

$$\psi(t) = c_n \sum_m \frac{[a + n - 1]_m}{|m|!} Z_m(t) = c_n \det(1 - t)^{-(a+n-1)} \tag{3.20}$$

for all $\|t\| < 1$. By (3.8)

$$\frac{\det(K_2(s_i, t_j))}{V(s)\, V(t)} = c_n \int_{U(n)} \det(1 - sutu^{-1})^{-(a+n-1)}\, du \tag{3.21}$$

with $\det(1 - sutu^{-1})^{-(a+n-1)} > 0$ for $\|s\| \cdot \|t\| < 1$. For $a > 0$ the constant c_n is positive, and it is obvious from (3.21) that $K_2(x, y) = (1 - xy)^{-a}$ is STP_∞ on $\{(x, y) \in \mathbf{R}^2\colon |xy| < 1\}$. In general, by (3.19), the sign of c_n agrees with that of $\prod_{j=1}^n (a)_{j-1}$.

The previous two examples, while highlighting the power and simplicity of the integral representation (3.8) in establishing total positivity, also illustrate the limitations of the presentation (3.6) of the functions ψ as spherical series. That is to say, in the above examples we must have the assurance that the functions ψ defined by the spherical series

$$\psi(t)=\sum_m \frac{1}{|m|!} Z_m(t) \tag{3.22}$$

and

$$\psi(t)=\sum_m \frac{[a+n-1]_m}{|m|!} Z_m(t) \tag{3.23}$$

are positive on the appropriate subsets of S. However, the positivity is not obvious until we sum the series (3.22) and (3.23) to the familiar closed form expressions $e^{\operatorname{tr}(t)}$ and $\det(1-t)^{-(a+n-1)}$, respectively.

In general, situations will arise in which we wish to apply the integral representation (3.8) in order to prove total positivity, but it is not a priori obvious that the integrand ψ, given by the spherical series (3.6), is non-negative. Extremely useful is such cases are integral formulas for the spherical functions Z_m, which present us with the opportunity to interchange an integral and an infinite sum in (3.6). In this way, the spherical series (3.6) can be "summed" to an integral representation for ψ. The hope, of course, is that the integrand in this integral representations for ψ is non-negative. Simply phrased, integral formulas for the spherical functions provide a means of applying Polya's principle a second time.

An example of such a "reproducing" integral formula is the *Euler integral* [5, Corollary 5.10]

$$Z_m(t)=\frac{\Gamma_n(b)}{\Gamma_n(b-a)\,\Gamma_n(a)}\,\frac{[b]_m}{[a]_m}\int_{0<r<1} Z_m(rt)\det(r)^{a-n}\det(1-r)^{b-a-n}\,dr \tag{3.24}$$

valid for $\operatorname{Re}(a)>n-1$, $\operatorname{Re}(b-a)>n-1$, and $t\in S_n$. The integration takes place over the (generalized) *unit interval* of Hermitian matrices r all of whose eigenvalues are between 0 and 1; dr is Lebesgue measure; and

$$\Gamma_n(a)=\pi^{n(n-1)/2}\prod_{i=1}^{n}\Gamma(a-i+1) \tag{3.25}$$

is the *gamma function associated to the space* S_n. For future reference, the reader is cautioned in (3.24) to note the dependence upon n of the constraints on the parameters a and b.

Another such integral formula involves the *highest weight vector* [5, paragraph 4.7]

$$q_m(t) = \det(t)^{m_n} \prod_{j=1}^{n-1} \Delta_j(t)^{m_j - m_{j-1}} \tag{3.26}$$

which is a polynomial function on S_n having certain specified transformation properties with respect to the upper triangular subgroup of $GL(n, \mathbf{C})$. The notation $\Delta_j(t)$ in (3.26) denotes the jth principal minor of the matrix t. Then [5, (4.8.2)]

$$Z_m(t) = d_m \omega_m \int_{U(n)} q_m(utu^{-1})\, du \tag{3.27}$$

for all $t \in S_n$. This formula can be put to good use (see Lemma 5.2) when the partition m has only one non-zero entry. Let us adopt the notation (j) for the partition $(j, 0, ..., 0)$. Then from (2.4) and (2.12),

$$d_{(j)} = \binom{n+j-1}{j} = \frac{(n)_j}{j!} \qquad \text{and} \qquad \omega_{(j)} = 1, \tag{3.28}$$

and by (3.26)

$$q_{(j)} = \Delta_1(t)^j = t_{11}^j \tag{3.29}$$

for all $t \in S_n$. Inserting (3.28) and (3.29) in (3.27), we obtain

$$Z_{(j)}(t) = \frac{(n)_j}{j!} \int_{U(n)} [(utu^{-1})_{11}]^j\, du \tag{3.30}$$

for all $t \in S_n$.

We close this section with the remark that the functions K_1 and K_2 in Corollaries 3.3 and 3.4 are classical hypergeometric functions ${}_0\mathscr{F}_0$ and ${}_1\mathscr{F}_0$, respectively. Similarly, the associated spherical series (3.22) and (3.23)—from which the total positivity of K_1 and K_2 is derived—are the *hypergeometric functions* ${}_pF_q$ *of matrix argument* corresponding to $(p, q) = (0, 0)$ and $(1, 0)$, respectively. The connection between hypergeometric functions of matrix argument and total positivity of classical hypergeometric functions is the subject of the remainder of the paper.

4. Hypergeometric Functions of Matrix Argument

The theory of hypergeometric functions of matrix argument is described in detail in [5, 6, 7, and 9]. Over the complex field such functions were originally defined for use in multivariate statistics by James [9], and their study was later taken up by other researchers in statistics, quantum physics, harmonic analysis, and number theory (cf. [4, 13, 15, 18, 19, and 20]). The special case (2.15), the binomial theorem, appears much earlier [8], due to its important role in function theory on the classical domains in several complex variables.

The *hypergeometric function* ${}_pF_q$ *of matrix argument* is defined to be the real-analytic function on S_n given by the spherical series

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; t) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{|m|=j} \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} Z_m(t), \quad (4.1) $$

where for $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant n$ none of the numbers $-b_i + j - 1$ is a non-negative integer. We use the symbol $\mathscr{F}$ to denote the classical hypergeometric function ${}_p\mathscr{F}_q(a_1, ..., a_p; b_1, ..., b_q; t)$ to which (4.1) reduces when $n = 1$.

In analogy to the classical context, from (2.14) and (2.15) we see that $(p, q) = (0, 0)$ corresponds to the exponential function ${}_0F_0(t) = e^{\mathrm{tr}(t)}$ and $(p, q) = (1, 0)$ to the binomial theorem ${}_1F_0(a; t) = \det(1 - t)^{-a}$ for $\|t\| < 1$. The case $(p, q) = (0, 1)$ defines functions ${}_0F_1(a; t)$ from which *Bessel functions of matrix argument* [4] are obtained in much the same way that classical Bessel functions J_a of the first kind are derived from ${}_0\mathscr{F}_1(a; t)$. The cases $(p, q) = (1, 1)$ and $(2, 1)$ correspond to the *confluent* and *Gaussian hypergeometric functions*, respectively. All of these hypergeometric functions satisfy analogs of formulas well known for their classical counterparts (e.g., (4.10) below).

Aware of the strong connection between total positivity and functions of *two* matrix arguments, as evidenced by the expansion (3.11), we define the *hypergeometric function of two matrix arguments*, also denoted ${}_pF_q$, by the series

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; s, t) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{|m|=j} \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} \frac{Z_m(s)\, Z_m(t)}{Z_m(1)}. \quad (4.2) $$

Such functions were originally defined and treated by James [9] in his statistical studies. The convergence properties of (4.2) are summarized in the following theorem, the proof of which is entirely analogous to that given in [5, Theorem 6.3] for the one-argument functions. Recall that for $t \in S$ we let $\|t\| = \max\{|t_i|: i = 1, ..., n\}$, where $t_1, ..., t_n$ are the eigenvalues of t.

THEOREM 4.1. (1) *If* $p \leqslant q$ *then the hypergeometric series* (4.2) *converges absolutely for all* s *and* t *in* S_n.

(2) *If* $p = q+1$ *then the series* (4.2) *converges absolutely for* $\|s\| \cdot \|t\| < 1$ *and diverges for* $\|s\| \cdot \|t\| > 1$.

(3) *If* $p > q+1$ *then the series* (4.2) *diverges unless it terminates.*

We can recapture the hypergeometric function (4.1) of one matrix argument from the two-argument functions by letting $s = 1$ in (4.2). Conversely, the hypergeometric function of two matrix arguments can be derived from the one-argument version by integration over $U(n)$. That is, by substituting (2.6) in (4.2) and applying Fubini's theorem, we obtain

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; s, t) = \int_{U(n)} {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; sutu^{-1})\, du. \tag{4.3}$$

Note that the special cases

$$ {}_0F_0(s, t) = \int_{U(n)} e^{\operatorname{tr}(sutu^{-1})}\, du \tag{4.4}$$

and

$$ {}_1F_0(a; s, t) = \int_{U(n)} \det(1 - sutu^{-1})^{-a}\, du \tag{4.5}$$

have already appeared in (3.15) and (3.21), respectively. Indeed, (3.15) and (3.21) can be rewritten, respectively, as

$$ \frac{\det({}_0\mathscr{F}_0(s_i t_j))}{V(s)\, V(t)} = \beta_n^{-1}\, {}_0F_0(s, t) \tag{4.6}$$

for all $s, t \in S_n$, and

$$ \frac{\det({}_1\mathscr{F}_0(a; s_i t_j))}{V(s)\, V(t)} = c_n\, {}_1F_0(a+n-1; s, t) \tag{4.7}$$

for $\|s\| \cdot \|t\| < 1$.

The theorem that follows is a generalization of Formulas (4.6) and (4.7) to all the hypergeometric functions ${}_pF_q$ of matrix argument. This result was proved independently by Gross and Richards [6], where two different proofs are given, and by Khatri [13].

THEOREM 4.2. *If the eigenvalues of s and t are denoted by* $s_1, ..., s_n$ *and* $t_1, ..., t_n$, *respectively, then*

$$
\begin{aligned}
&{}_pF_q(a_1, ..., a_p; b_1, ..., b_q; s, t) \\
&= c_{p,q} \frac{\det({}_p\mathscr{F}_q(a_1-n+1, ..., a_p-n+1; b_1-n+1, ..., b_q-n+1; s_it_j))}{V(s)\,V(t)},
\end{aligned} \tag{4.8}
$$

where

$$
c_{p,q} = \beta_n^{-1} \frac{\prod_{i=1}^{n} \prod_{j=1}^{q} (b_j - n + 1)_{n-i}}{\prod_{i=1}^{n} \prod_{j=1}^{p} (a_j - n + 1)_{n-i}}. \tag{4.9}
$$

Some remarks on Theorem 4.2 are in order. One of our proofs in [6] proceeds by induction on p and q, and relies on the *Euler formula*

$$
\begin{aligned}
&{}_{p+1}F_{q+1}(a_1, ..., a_{p+1}; b_1, ..., bb_{q+1}; s, t) = \frac{\Gamma_n(b_{q+1})}{\Gamma_n(a_{p+1})\,\Gamma_n(b_{q+1}-a_{p+1})} \\
&\times \int_{0<r<1} \det(r)^{a_{p+1}-n} \det(1-r)^{b_{q+1}-a_{p+1}-n}\, {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; rs, t)\,dr
\end{aligned} \tag{4.10}
$$

valid for $\operatorname{Re}(a_{p+1}) > n-1$, $\operatorname{Re}(b_{q+1} - a_{p+1}) > n-1$. Formula (4.10) is derived by expanding the ${}_pF_q$ in the integrand in its spherical series (4.4) and applying (3.24) to integrate term-by-term. The other proof in [6] and the proof in [13] both use a variant of the Binet–Cauchy formula [10, p. 1]. The special case $(p, q) = (2, 1)$ and $s = 1$ was also proved in [15]. The result and proof in [13] have been widely overlooked, possibly because the review [17] in *Mathematical Reviews* provides only the title of the paper and no information on its content. In our case, we were led to conceive of Theorem 4.2 from the ingenious techniques of L.-K. Hua [8] on spherical series. In particular, one of his results [8, (1.2.5)] is equivalent to the special case of (4.8) in which $(p, q) = (0, 1)$ and $a_1 = 1$. This result led us directly to the formulation and proof of the general theorem. Only recently, in a literature search, did we discover that the result had been proved earlier in [13].

In [6] we were guided by the philosophy that a formula such as (4.8) could be used to deduce new information about the hypergeometric functions of matrix argument from well-known properties of the classical hypergeometric functions. Here, we have the inverse situation in which we utilize the hypergeometric functions of two matrix arguments to study the total positivity of the classical hypergeometric functions. In particular, if we let

$f(x) = {}_p\mathscr{F}_q(a_1, ..., a_p;\ b_1, ..., b_q;\ x)$ in Theorem 3.1, then substituting (4.3) and (4.8) into (3.8) immediately gives us the following result.

COROLLARY 4.3. *Let* $f(x) = {}_p\mathscr{F}_q(a_1, ..., a_p;\ b_1, ..., b_q; x)$, *and set* $K(x, y) = f(xy)$. *Then for each* $n = 1, 2, ...$,

$$\frac{\det(K(s_i, t_j))}{V(s)\,V(t)} = c_{p,q}^{-1}\ {}_pF_q(a_1 + n - 1, ..., a_p + n - 1; b_1 + n - 1, ..., b_q + n - 1; s, t), \quad (4.11)$$

or equivalently,

$$\frac{\det(K(s_i, t_j))}{V(s)\,V(t)} = c_{p,q}^{-1} \int_{U(n)} {}_pF_q(a_1 + n - 1, ..., a_p + n - 1; b_1 + n - 1, ..., b_q + n - 1; sutu^{-1})\, du. \quad (4.12)$$

In particular, the classical hypergeometric function $K(x, y) = {}_p\mathscr{F}_q(a_1, ..., a_p;\ b_1, ..., b_q;\ xy)$ *will be* TP_∞ *if and only if the hypergeometric functions*

$$\psi_n(t) = {}_pF_q(a_1 + n - 1, ..., a_p + n - 1; b_1 + n - 1, ..., b_q + n - 1; t) \quad (4.13)$$

of matrix argument are non-negative for all positive integers n.

If we utilize the connection between ETP and STP_∞ (see (1.2) *et seq.*), then we can rephrase the criterion of Corollary 4.3 in an apparently stronger form.

COROLLARY 4.4. *The classical hypergeometric function* $K(x, y) = {}_p\mathscr{F}_q(a_1, ..., a_p;\ b_1, ..., b_q; xy)$ *is ETP (and a fortiori,* STP_∞*) if and only if* $\psi_n(x1) = {}_pF_q(a_1 + n - 1, ..., a_p + n - 1;\ b_1 + n - 1, ..., b_q + n - 1;\ x1) > 0$ *for all* $n = 1, 2, ...$

On the surface, it would seem that the criterion for ETP (Corollary 4.4) is perhaps easier to apply than the direct STP_∞ condition (Corollary 4.3) since the former involves a function of just one variable. However, in general, the power series

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x1) = \sum_{j=0}^{\infty} \left(\sum_{|m|=j} \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} \frac{d_m^2}{[n]_m} \right) x^j \quad (4.14)$$

for that function is quite complicated, even in the simplest non-classical case $n=2$. For this reason, we must go further into the structure of matrix argument hypergeometric functions in order to pursue total positivity properties.

5. The Confluent and Gaussian Hypergeometric Functions

Having already treated the hypergeometric functions ${}_0\mathscr{F}_0$ and ${}_1\mathscr{F}_0$ functions, in this section we apply Corollary 4.3 to establish total positivity properties of the *confluent hypergeometric function* ${}_1\mathscr{F}_1$ and the *Gaussian hypergeometric function* ${}_2\mathscr{F}_1$. We make some preliminary observations concerning the corresponding hypergeometric functions of matrix argument.

First, by (4.1),

$$ {}_1F_1(a;b;t)=\sum_m \frac{[a]_m}{[b]_m}\frac{Z_m(t)}{|m|!} \tag{5.1}$$

for $t\in S_n$, where b is not an integer less than n; and

$$ {}_2F_1(a,b;c;t)=\sum_m \frac{[a]_m\,[b]_m}{[c]_m}\frac{Z_m(t)}{|m|!} \tag{5.2}$$

for $\|t\|<1$, where c is not an integer less than n. The Euler integral (4.10) for ${}_1F_1$ takes the form

$$ {}_1F_1(a;b;t)=\frac{\Gamma_n(b)}{\Gamma_n(a)\,\Gamma_n(b-a)}\int_{0<r<1} e^{\operatorname{tr}(rt)}\det(r)^{a-n}\det(1-r)^{b-a-n}\,dr \tag{5.3}$$

for all $t\in S_n$, with convergence for $\operatorname{Re}(a)>n-1$ and $\operatorname{Re}(b-a)>n-1$. Similarly,

$$ {}_2F_1(a,b;c;t) = \frac{\Gamma_n(c)}{\Gamma_n(b)\,\Gamma_n(c-b)}\int_{0<r<1}\det(r)^{b-n}\det(1-r)^{c-b-n}\det(1-rt)^{-a}\,dr \tag{5.4}$$

for all $t\in S_n$ such that $\|t\|<1$, with convergence for $\operatorname{Re}(b)>n-1$ and $\operatorname{Re}(c-b)>n-1$. Since the integrand in both Euler formulas is positive whenever a, b, and c are real, Corollary 4.3 immediately implies the following sufficient conditions for total positivity.

THEOREM 5.1. (1) *Let* $a>0$ *and* $b-a>n-1$. *Then* $K(x, y)={}_1\mathscr{F}_1(a; b; xy)$ *is* STP_n *on* $\mathbf{R}^2$.

(2) *Let* $b>0$ *and* $c-b>n-1$. *Then* $K(x, y)={}_2\mathscr{F}_1(a, b; c; xy)$ *is* STP_n *on the domain* $\{(x, y)\in\mathbf{R}^2\colon |xy|<1\}$.

Next, we establish the *Kummer identities*

$$ {}_1F_1(a; b; t)=e^{-\operatorname{tr}(t)}\,{}_1F_1(b-a; b; -t) \tag{5.5}$$

for $t\in S_n$, and

$$ {}_2F_1(a, b; c; t)=\det(1-t)^{c-a-b}\,{}_2F_1(c-a, c-b; c; t) \tag{5.6}$$

for $\|t\|<1$. Both (5.5) and (5.6) are valid without restriction on a, b, or c. To prove (5.5) in the range $\operatorname{Re}(a)>n-1$ and $\operatorname{Re}(b-a)>n-1$, make the change of variable $r\to 1-r$ in (5.3). In general, (5.5) follows for all a and b by analytic continuation. Similarly, if $\operatorname{Re}(b)>n-1$ and $\operatorname{Re}(c-b)>n-1$, then by the same change of variables in (5.4), we obtain

$$ {}_2F_1(a, b; c; t)=\det(1-t)^{-a}\,{}_2F_1(a, c-b; c; -t(1-t)^{-1}), \tag{5.7}$$

and by symmetry in the numerator parameters,

$$ {}_2F_1(a, b; c; t)=\det(1-t)^{-a}\,{}_2F_1(c-b, a; c; -t(1-t)^{-1}) \tag{5.8}$$

for $\|t\|<1$ and $\|t(1-t)^{-1}\|<1$. By analytic continuation, (5.8) holds for all a, b, and c. Now, (5.6) follows by applying (5.7) to the right side of (5.8).

Finally, we relate the matrix argument hypergeometric functions having some numerator parameter equal to one to the classical hypergeometric functions. The form in which we use this result is as follows.

LEMMA 5.2. *Let* n *be any positive integer and* $a, b\in\mathbf{C}$. *Then*

$$ {}_1F_1(1; a+n; t)=\int_{U(n)} {}_1\mathscr{F}_1(n; a+n; (utu^{-1})_{11})\,du \tag{5.9}$$

for $t\in S_n$, *and*

$$ {}_2F_1(1, a-b+1; a+n; t)=\int_{U(n)} {}_2\mathscr{F}_1(n, a-b+1; a+n; (utu^{-1})_{11})\,du \tag{5.10}$$

for all $\|t\|<1$.

Proof. Observe that for any partition $m=(m_1, ..., m_n)$,

$$[1]_m=\begin{cases} m_1!, & \text{if } m_2=0 \\ 0, & \text{if } m_2>0. \end{cases} \tag{5.11}$$

Thus, substituting (3.28) and (3.30) into (5.1) and applying Fubini's theorem, we obtain

$$\begin{aligned}
{}_1F_1(1; a+n; t) &= \sum_m \frac{[1]_m}{[a+n]_m} \frac{Z_m(t)}{|m|!} = \sum_{j=0}^{\infty} \frac{Z_{(j)}(t)}{(a+n)_j} \\
&= \sum_{j=0}^{\infty} \frac{(n)_j}{j!(a+n)_j} \left(\int_{U(n)} [(utu^{-1})_{11}]^j \, du \right) \\
&= \int_{U(n)} \left(\sum_{j=0}^{\infty} \frac{(n)_j}{j!(a+n)_j} [(utu^{-1})_{11}]^j \right) du \\
&= \int_{U(n)} {}_1\mathscr{F}_1(n; a+n; (utu^{-1})_{11})_{du}
\end{aligned}$$

for all $t \in S_n$. In like fashion, (5.2) becomes

$$\begin{aligned}
{}_2F_1(1, a-b+1; a+n; t) &= \sum_m \frac{[1]_m [a-b+1]_m}{[a+n]_m} \frac{Z_m(t)}{|m|!} \\
&= \sum_{j=0}^{\infty} \frac{(a-b+1)_j Z_{(j)}(t)}{(a+n)_j} \\
&= \sum_{j=0}^{\infty} \frac{(a-b+1)_j (n)_j}{j!(a+n)_j} \left(\int_{U(n)} [(utu^{-1})]^j \, du \right) \\
&= \int_{U(n)} \left(\sum_{j=0}^{\infty} \frac{(a-b+1)_j (n)_j}{j!(a+n)_j} [(utu^{-1})_{11}]^j \right) du \\
&= \int_{U(n)} {}_2\mathscr{F}_1(n, a-b+1; a+n; (utu^{-1})_{11}) \, du
\end{aligned}$$

for all $\|t\| < 1$.

THEOREM 5.3. *Let* $K_{1,1}(x, y) = {}_1\mathscr{F}_1(a; a+1; xy)$ *and* $K_{2,1}(x, y) = {}_2\mathscr{F}_1(a, b; a+1; xy)$.

(1) *If* $a>0$, *then* $K_{1,1}$ *is* STP_∞ *for all* $(x, y) \in \mathbf{R}^2$.

(2) *If* $a>0$ *and* $b>0$, *then* $K_{2,1}$ *is* STP_∞ *for all* $(x, y) \in \mathbf{R}^2$ *such that* $|xy|<1$.

The results in Theorem 5.3 were originally proved by Burbea [3] via more classical methods. However, in the case of the Gaussian hypergeometric function our methods yield a larger domain of total positivity.

Proof of Theorem 5.3. Let $a, b > 0$. We apply Corollary 4.3, first with $p = q = 1$, then with $p = 2$ and $q = 1$.

In the former case, $f(xy) = K_{1,1}(x, y)$. By (4.13), we need only show that for each n the function $\psi_n(t) = {}_1F_1(a+n-1;\ a+n;\ t)$ is positive for all $t \in S_n$; or by the Kummer identity (5.5), that ${}_1F_1(1; a+n; t) > 0$ for all $t \in S_n$. One would like to say that this result is obvious from the Euler formula (5.3) with $a = 1$ and $b = a+n$. Unfortunately, since $a = 1$, the condition $\mathrm{Re}(a) > n-1$ for the validity of (5.3) is not satisfied for $n \geqslant 2$. Hence, we invoke Lemma 5.2. By (5.9), the positivity of ${}_1F_1(1;\ a+n;\ t)$ is reduced to that of the classical confluent hypergeometric function ${}_1\mathscr{F}_1$. However, since ${}_1\mathscr{F}_1$ is classical (i.e., $n = 1$), the Euler formula (5.3) holds for the integrand in (5.9) and implies that the integrand is positive. Therefore, by (5.9), ${}_1F_1(1;\ a+n;\ t) > 0$ for all $t \in S_n$.

The latter case is proved in the same way. Here, $f(xy) = K_{2,1}(x, y)$ and $\psi_n(t) = {}_2F_1(a+n-1,\ b+n-1;\ a+n;\ t)$ for $\|t\| < 1$. By the Kummer identity (5.6), it is enough to show that ${}_2F_1(1,\ a-b+1;\ a+n; t) > 0$ for $\|t\| < 1$. By the Euler formula (5.4) for the classical case $n = 1$, the integrand in (5.10) is positive. Thus, ${}_2F_1(1,\ a-b+1;\ a+n;\ t)$ is also positive.

Remarks 5.4. (1) A variant of the preceding proof of Theorem 5.3 can be based upon Corollary 4.4 and the Kummer identities (5.5) and (5.6). For example, if ${}_1F_1(1;\ a+n;\ x1) > 0$ for all $x \in \mathbf{R}$ and all n, then ${}_1\mathscr{F}_1(a;\ a+1;\ xy)$ is ETP. For scalar arguments, however, the integral formula (5.9) has the simpler form

$$ {}_1F_1(1; a+n; x1) = {}_1\mathscr{F}_1(n; a+n; x) \tag{5.12} $$

for all $x \in R$. More generally, from (4.14) and (3.28)

$$ {}_pF_q(1, a_2, ..., a_p; b_1, ..., b_q; x1) = {}_p\mathscr{F}_q(n, a_2, ..., a_p; b_1, ..., b_q; x) \tag{5.13} $$

for all $x \in \mathbf{R}$. This is one instance when a special case of (4.14) may be directly related to a classical hypergeometric series.

(2) There is another method for proving the positivity of ${}_1F_1(1; a+n; t)$, $a > 0$, that relies on an explicit formula [5, Lemma 6.8] for the polynomials $Z_{(j)}(t)$. This approach leads to an interesting Dirichlet-type integral

$$ {}_1F_1(1; a+n; t) = (a)_n \int_{\Sigma_n} \left(1 - \sum_{i=1}^{n} x_i\right)^{a-1} \exp\left(\sum_{i=1}^{n} t_i x_i\right) dx_1 \cdots dx_n \tag{5.14} $$

where $\Sigma_n = \{(x_1, ..., x_n): x_i > 0, i = 1, ..., n; \sum_{i=1}^n x_i < 1\}$. The integral (5.14) is clearly positive for all $t \in S_n$.

(3) Finally, there is still another proof of Theorem 5.3. This proof does not use the Kummer identities. Instead, it proceeds by an inductive step from ${}_0\mathscr{F}_0$ and ${}_1\mathscr{F}_0$ to ${}_1\mathscr{F}_1$ and ${}_2\mathscr{F}_1$, respectively. It is based upon applying the Binet–Cauchy formula to the Euler integral (4.10) with $n = 1$. The general induction step offers no difficulty, and gives the following results, first proved by Burbea [3]. Except for an enlargement of the domain of total positivity for the ${}_{p+1}\mathscr{F}_p$ function, the proof is similar to that in [3, Theorems 6 and 7].

THEOREM 5.5. (1) *Let $a_i > 0$ for $i = 1, ..., p$. Then, the function $K(x, y) = {}_p\mathscr{F}_p(a_1, ..., a_p; a_1 + 1, ..., a_p + 1; xy)$ is STP_∞ on $\mathbf{R}^2$.*

(2) *Let $a_i > 0$ for $i = 0, 1, ..., p$. Then the function $K(x, y) = {}_{p+1}\mathscr{F}_p(a_0, a_1, ..., a_p; a_1 + 1, ..., a_p + 1; xy)$ is STP_∞ on $\{(x, y) \in \mathbf{R}^2: |xy| < 1\}$.*

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